

$$\vartheta^7 = 1 + \frac{256}{15} \left\{ \sum_{1,3,5,\dots} \frac{(-1)^{\frac{1}{2}(k-1)}}{k^3} \sum_j \sum_{m=0}^{\infty} (mk+j)^{\frac{3}{2}} q^{mk+j} \right. \\ \left. - \sum_{2,4,6,\dots} \frac{1}{k^2} \sum_j \sum_{m=0}^{\infty} (-1)^{m+\mu} (mk+j)^{\frac{3}{2}} q^{mk+j} \right\}, \quad (11)$$

$$\vartheta^3 = 1 + 8 \left\{ \sum_{1,3,5,\dots} \frac{(-1)^{\frac{1}{2}(k-1)}}{k} \sum_j \sum_{m=0}^{\infty} (mk+j)^{\frac{1}{2}} q^{mk+j} \right. \\ \left. + \sum_{2,4,6,\dots} \frac{1}{k} \sum_j \sum_{m=0}^{\infty} (-1)^{m+\mu} (mk+j)^{\frac{1}{2}} q^{mk+j} \right\}. \quad (12)$$

The interpretation of j and μ is as before, except that, when k is even, j is a residue of one of the numbers $\frac{1}{2}k, \frac{1}{2}k + 1^2, \dots, \frac{1}{2}k + (k-1)^2$. These identities embody the theory for 7 or 3 squares. It should be noted however, that the application of my method becomes very much more difficult when $s = 3$, as the double series used are then not absolutely convergent; and in fact the only proof of (12) which I possess consists in an identification of the results which it gives with those already known.

I conclude by a word concerning the cases in which $s > 8$. Here, when s is odd, we are on untrodden ground. We have the asymptotic formula (6); and we can evaluate $X_s(n)$ as when $s = 5$ or 7, thus obtaining a series of new results. But it is no longer to be expected that our results should be *exact*, and I have verified that, when $s = 9$, they are not exact, even when $n = 1$.

¹ Glaisher, J. W. L., *Proc. London Math. Soc.*, (Ser. 2), 5, 1907, (479-490).

² Ramanujan, S., *Trans. Camb. Phil. Soc.*, 22, 1916, (159-184); *Ibid.*, (in course of publication).

³ Mordell, L. J., *Quart. J. Math.*, 48, 1917, (93-104).

⁴ Hardy, G. H., and Ramanujan, S., *Proc. London Math. Soc.*, (Ser. 2), 17, 1918, (in course of publication).

THE CRYSTAL STRUCTURE OF ICE

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During the winter of 1916-1917 the crystal structure of ice was investigated by means of the X-rays. The photographic method originated by deBrogli¹ was used with certain modifications suggested privately by Dr. A. W. Hull. The source of energy was a Coolidge tube with tungsten target excited by an induction coil with mercury turbine interrupter. At first the

apparatus already set up for another investigation was used, being kept cool by leaving the laboratory windows open. This method was very uncertain on account of the erratic weather and was otherwise unsatisfactory and was discarded after a single good photograph had been obtained. The spectrometer system was then enclosed and the chamber kept cool by cans of ice and salt. By this means the temperature could be kept reasonably constant but it was found virtually impossible to mount and maintain a specimen long enough to get a satisfactory photograph, probably on account of the presence of the salt vapor. Upon the recommendation of Prof. A. G. Webster a grant was made from the Rumford Fund of the American Academy of Arts and Sciences, Boston, in aid of the investigation which made it possible to install a small ammonia refrigerating machine loaned by the Automatic Refrigerating Company and to build a specially adapted spectrometer mounted in a well-insulated refrigerator box. With this equipment the temperature could be maintained indefinitely and there was no further trouble from melting of specimens. A marked tendency to sublimation, however, was troublesome until each specimen was mounted in a gelatine capsule when equilibrium was quickly established between the crystal and its vapor. Protected in this manner specimens were preserved for days.

Commercial artificial ice was first investigated as it shows marked prismatic structure. Unfortunately the prisms are distorted through pressure in the formation of the ice so that it is difficult to identify cleavage planes. Some photographs show spectral bands but no sharp lines upon which to base calculations. The investigation is to be pursued further, however, as it is probable that sufficiently small crystals or slices of crystals will give sharply defined lines. In a second procedure a thin layer of ice, about 2 mm. thick, was allowed to form on a pan of tap water. It was difficult to identify individual crystals and more difficult to isolate them for mounting but occasionally a reasonably good specimen was secured and mounted so that the axis of rotation bore a definite relation to the original surface of the ice. Orientation with respect to other axes was a matter of guess work, usually wrong as the results showed. Nevertheless several satisfactory photographs were obtained from specimens prepared in this manner, in fact the calculations are based entirely on them. In a third method ice was frozen out of a weak salt solution. In this way a large crop of thin specimens showing distinct cleavage planes could be secured. Owing to the failure of the source of power the investigation was interrupted at this point and has not yet been renewed. One photograph was obtained showing distinct spectral bands but no identifiable lines, possibly on account of the mixture of microscopic salt crystals with the ice. The method is promising and is to be pursued further at a convenient season.

Ice is commonly assigned to the hexagonal system of crystals² and is conveniently referred to a triangular space lattice each cell of which has sides

a and a height c . An elementary triangular cell of such a structure will have a volume $V = \sqrt{3}a^2c/4$. If the density is ρ , the mass of a molecule m and the number of molecules per cell n , the mass of the cell is $nm = 3a^2c\rho/4$. For ice² $c/a = 1.4026$, $\rho = .91$ gm/cm³, $m = 29.73 \times 10^{-24}$ gm. The molecular weight is taken, as the arrangement of diffracting centers is fundamentally that of the molecule, each pair of hydrogen atoms being presumably near an oxygen atom. These values gave

$$a = (54.35n)^{\frac{1}{3}} \times 10^{-8} \text{ cm.}$$

For triangular lattices the following values of n occur: simple lattice, $n = \frac{1}{2}$; two interpenetrating lattices, $n = 1$; three interpenetrating lattices, $n = \frac{3}{4}$ or $(\frac{3}{4})$; four interpenetrating lattices, $n = 2$.

Values of a , $a/2$, (the spacing of the $\bar{1}\bar{2}10$ planes), $a\sqrt{3}/2$ (the spacing of the $10\bar{1}0$ planes and ca (the height of a cell) have been computed and are given in columns 2, 3, 4 and 5 of table 1. When h is the distance of the plate

TABLE 1

1	2	3	4	5	6	7	8	9	10	11
n	$\times 10^{-8} \text{ cm.}$				$h = 19.00 \text{ cm.}$			$h = 14.75 \text{ cm.}$		
	a	$a/2$	$a\frac{\sqrt{3}}{2}$	ca	$x_{11\bar{1}0}$	$x_{10\bar{1}0}$	x_{0001}	$x_{11\bar{1}0}$	$x_{10\bar{1}0}$	x_{0001}
$\frac{1}{2}$	3.01	1.50	2.60	4.22	2.67	1.54	0.95	2.07	1.20	0.74
$\frac{3}{4}$	3.44	1.72	2.98	4.83	2.33	2.35	0.83	1.81	1.04	0.65
1	3.79	1.89	3.38	5.32	2.11	1.18	0.76	1.65	0.92	0.59
2	4.33	2.16	3.75	6.08	1.85	1.07	0.66	1.44	0.83	0.51

from the axis of rotation, x the distance of a given line from the undeviated central image, d the distance between planes in the crystal, λ the wave-length of the radiation and N the order of the spectrum $x/h = N\lambda/d$, giving the relations $d = N\lambda h/x$ and $x = N\lambda h/d$. The wave-length used was the K line of tungsten $\lambda = .211 \times 10^{-8}$ cm. In certain cases h was 19.00 cm., in others 14.75 cm. Values of x corresponding to these values have been calculated for the three fundamental spacings of each of the forms having values of n already given. They are shown in columns 6 to 11 of table 1. The values of x determined from the four plates used in the calculations and the corresponding values of a are given in tables 2 to 5. The average value of a is 4.74×10^{-8} cm. indicating four interpenetrating lattices. From ~~table~~ 4 it appears that the 0001 spacing is $c/2$, i.e., the four sets of basal planes occur in pairs. A number of plausible models having such arrangement exist. They may be differentiated by the spacings of the pyramidal planes. It may be shown that in interpenetrating triangular lattices pyramids having indices of the form $(n\bar{O}\bar{p})$ have spacings according to the relation

$$d_{n\bar{O}\bar{p}} = K_2^a \sqrt{3} \sin. \phi_{n\bar{O}\bar{p}},$$

where ϕ_{nOnp} is the angle between the planes and the basal pinacoid 0001 and K is a factor varying from pyramid to pyramid in the same model and with different models for the same pyramid, its value depending upon the grouping of the planes in a given form. Values of this quantity have been

TABLE 2
 $h = 19.00$ cm.

INDEX	z	$d \times 10^8$ cm.	$a \times 10^8$ cm.
10 $\bar{1}0$	0.97	4.13	4.77
	1.98 (2)	4.05	4.68
	2.93 (3)	4.11	4.74
11 $\bar{2}0$	1.70	2.36	4.72
	3.40 (2)	2.36	4.72
01 $\bar{1}0$	0.95	4.22	4.87
	1.92 (2)	4.18	4.82
	2.90 (3)	4.15	4.79
$\bar{2}110$	1.70	2.36	4.72
	3.40 (2)	2.36	4.72
$\bar{1}010$	1.95 (2)	4.12	4.76
Average.....			4.76 \pm 0.04

TABLE 4
 $h = 14.75$ cm.

INDEX	z	$d \times 10^8$ cm.	$a \times 10^8$ cm.
0001	0.93	3.25	4.78
	1.92 (2)	3.24	4.82
01 $\bar{1}0$	1.50 (2)	4.15	4.79
	2.25 (3)	4.15	4.79
10 $\bar{1}2$	1.22	2.55	4.68
	2.45 (2)	2.54	4.67
30 $\bar{3}4$	2.90	1.07	4.81
11 $\bar{2}0$	2.65 (2)	2.34	4.68
30 $\bar{3}2$	4.5-5.2	1.19-1.38	
20 $\bar{2}1$	4.80 (2)	1.30	4.72
30 $\bar{3}1$	4.0-4.5	0.69-0.78	
40 $\bar{4}1$	5.5-6.0	1.04-1.13	
Average.....			4.74 \pm 0.06

TABLE 3
 $h = 14.75$ cm.

INDEX	z	$d \times 10^8$ cm.	$a \times 10^8$ cm.
$\bar{2}110$	1.34	2.32	4.64
	2.70 (2)	2.31	4.62
10 $\bar{1}0$	2.20 (3)	4.25	4.91
	3.00 (4)	4.15	4.79
$\bar{1}100$	1.50 (2)	4.15	4.79
$\bar{2}110$	1.35	2.31	4.62
Average.....			4.73 \pm 0.10

TABLE 5
 $h = 14.75$ cm.

INDEX	z	$d \times 10^8$ cm.	$a \times 10^8$ cm.
0001	1.85 (2)	3.38	4.78
	2.80 (3)	3.34	4.76
10 $\bar{1}1$	1.5-2.0	3.1-4.2	
	5.40 (6)	3.46	4.70
10 $\bar{1}2$	2.40 (2)	2.58	4.75
10 $\bar{1}3$	1.60	1.94	4.72
30 $\bar{3}2$	2.50	1.24	4.70
Average.....			4.72 \pm 0.04

determined by inspection of the models and are tabulated in table 6. The data of table 4 are consistent only with the distances calculated for model IV, which is therefore taken to represent the structure. In the tabulations but four models have been considered as these are all that satisfy reasonable considerations of symmetry.

The investigation shows that ice is properly assigned to the hexagonal system, that it consists of four interpenetrating triangular lattices, and that the fundamental spacings are

$$a = 4.74 \times 10^{-8} \text{ cm.}; h = 6.65 \times 10^{-8} \text{ cm.}$$

$$d_{11\bar{2}0} = 3.79 \times 10^{-8} \text{ cm.}; d_{10\bar{1}0} = 2.37 \times 10^{-8} \text{ cm.}; d_{0001} = 3.32 \times 10^{-8} \text{ cm.}$$

The arrangement of the lattices is conveniently explained by referring the origin of each lattice to two unit axes making an angle of 120° and a third

TABLE 6
VALUES OF K AND CORRESPONDING APPROXIMATE VALUES OF x WHEN $h = 14.75 \text{ cm.}$

INDEX	ϕ	Sin ϕ	MODEL I		MODEL II		MODEL III		MODEL IV	
			K	x	K	x	K	x	K	x
1011	$58^\circ 18' 30''$	0.8509	$\frac{1}{2}$	1.8	1	0.9	1	0.9	1	0.9
1012	$39^\circ 9' 9''$	0.6293	1	1.2	1	1.2	1	1.2	1	1.2
1013	$28^\circ 22' 0''$	0.4751	$\frac{1}{2}$	3.1	1	1.6	1	1.6	1	1.6
2021	$72^\circ 50' 20''$	0.9555	$\frac{1}{2}$	3.2	$\frac{1}{2}$	1.6	$\frac{1}{2}$	3.2	$\frac{1}{2}$	2.4
3031	$78^\circ 22' 13''$	0.9795	$\frac{1}{2}$	4.5	$\frac{1}{2}$	2.2	$\frac{1}{2}$	4.5	$\frac{1}{2}$	4.5
3032	$67^\circ 37' 36''$	0.9248	$\frac{1}{2}$	2.6	$\frac{1}{2}$	2.6	$\frac{1}{2}$	2.6	$\frac{1}{2}$	2.6
3034	$50^\circ 32' 15''$	0.7720	$\frac{1}{2}$	2.9	$\frac{1}{2}$	2.9	$\frac{1}{2}$	2.9	$\frac{1}{2}$	2.9
4041	$81^\circ 13' 30''$	0.9886	$\frac{1}{2}$	6.6	$\frac{1}{2}$	2.2	$\frac{1}{2}$	3.3	$\frac{1}{2}$	3.0

mutually perpendicular to these. The coordinates of the origins are then

$$0,0,0; \frac{1}{3}, \frac{2}{3}, 1/z; \frac{1}{3}, \frac{2}{3}, \frac{1}{2}; 0,0, (z+2)/2z.$$

The values of z , that is the relative displacement of the two planes making up a basal pair, is uncertain and needs further investigation. This requires a careful determination of the relative intensities of the spectra of different orders reflected from the 0001 planes and was beyond the scope of the present investigation. Conditions of symmetry suggest a value of $z = 6$, but this is purely conjectural.

The investigation was pursued under the direction of Prof. A. G. Webster of Clark University in the Physics Laboratory of the Worcester Polytechnic Institute.

¹ *Paris, C.-R. Acad., Sci.*, 157, Nov. 17, 1913, (924-926).

² *Dana, System of Mineralogy*, 1888 ed., p. 205.

FRINGING REEFS OF THE PHILIPPINE ISLANDS.

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A series of large-scale charts recently published by the United States Coast and Geodetic Survey for certain parts of the Philippine islands are, apart from their value to commerce, of much scientific interest in connection with